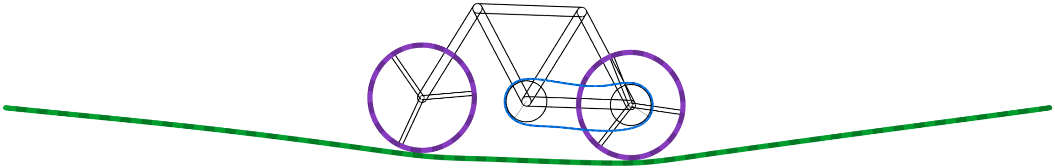
Efficient 2d rod model with frictional contact

Emile Hohnadel

April 14th 2026



What I did during my thesis

[Crespel et al. 2024]

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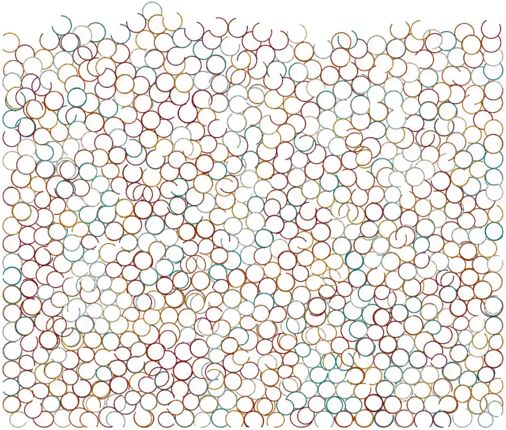
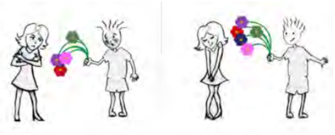
[Crespel et al. 2024]



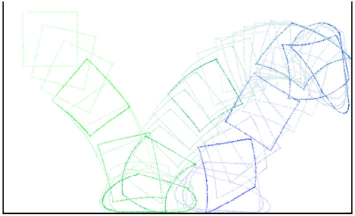
[Sano et al. 2023]

Utility of fast 2d rod model

Expressive animation



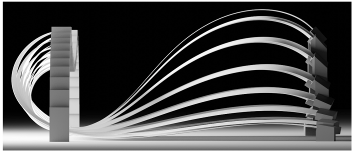
[Bertails-Descoubes 2012]



[Dvoroznak et al. 2017]

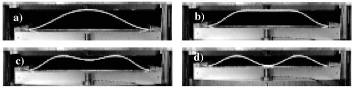
Fabrication

[Miguel et al. 2016]

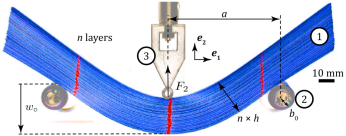


[Hafner & Bickel 2021]

Mechanical simulation

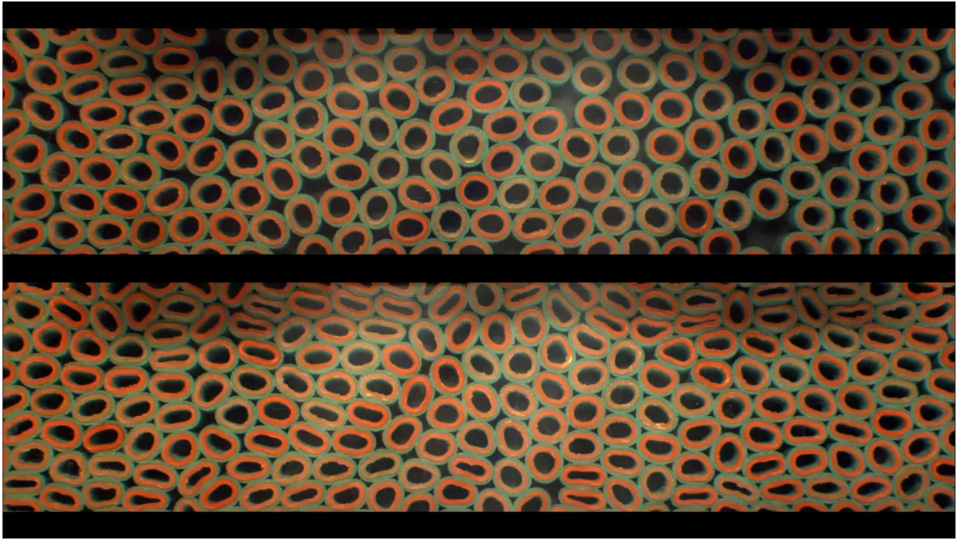


[Roman & Pocheau 1999]



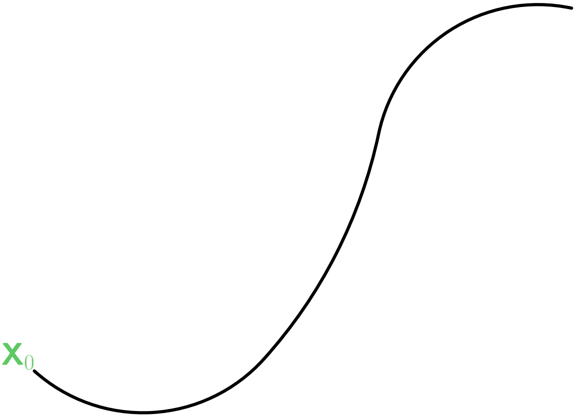
[Poincloux et al. 2021]

An impossible generalisation?



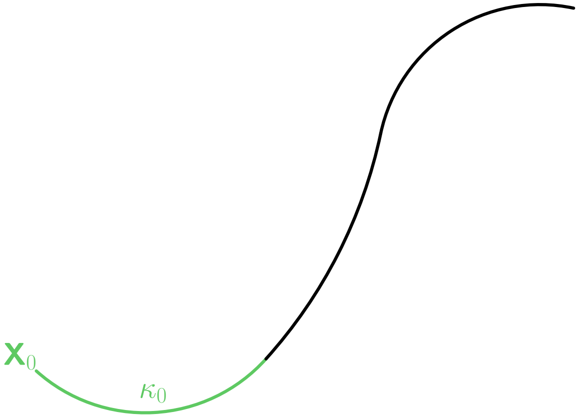
[Poincloux et al. 2024]

Chained super-circle model



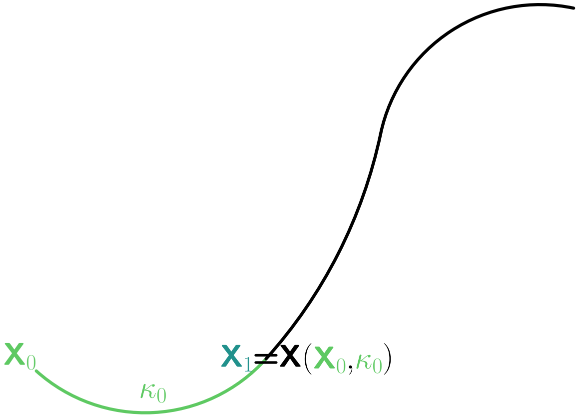
$$\mathbf{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix},$$

Chained super-circle model



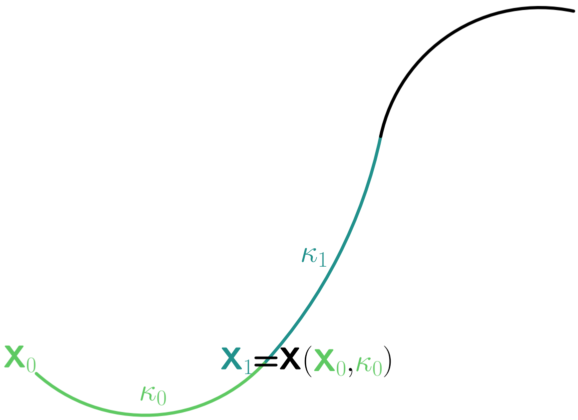
$$\mathbf{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix}, \ \kappa_0,$$

Chained super-circle model



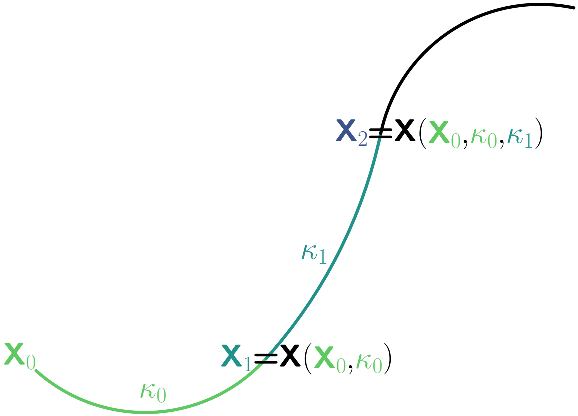
$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix}, \ \kappa_0, \ \boldsymbol{X}_1(\boldsymbol{X}_0, \kappa_0),$$

Chained super-circle model



$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix}, \ \kappa_0, \ \boldsymbol{X}_1(\boldsymbol{X}_0, \kappa_0), \ \kappa_1,$$

Chained super-circle model



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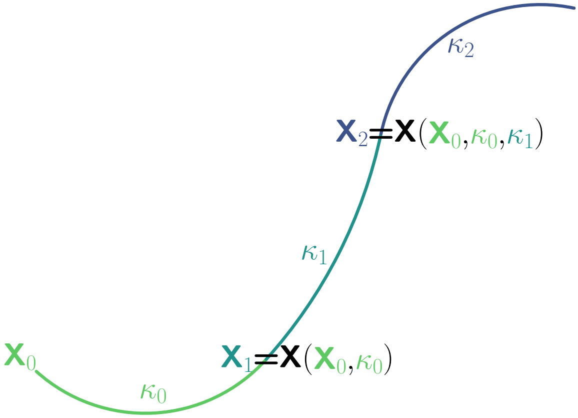
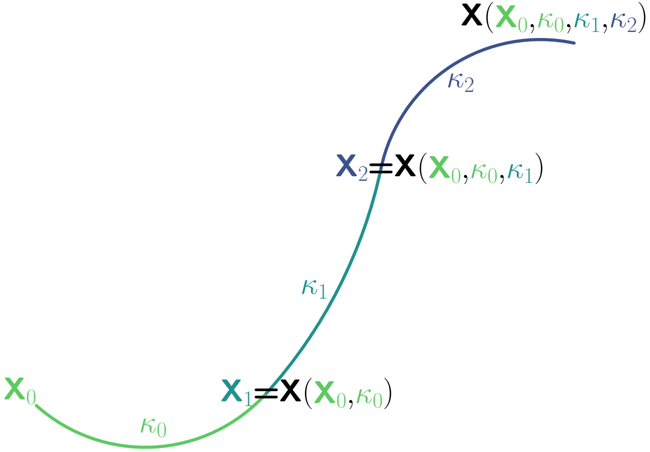


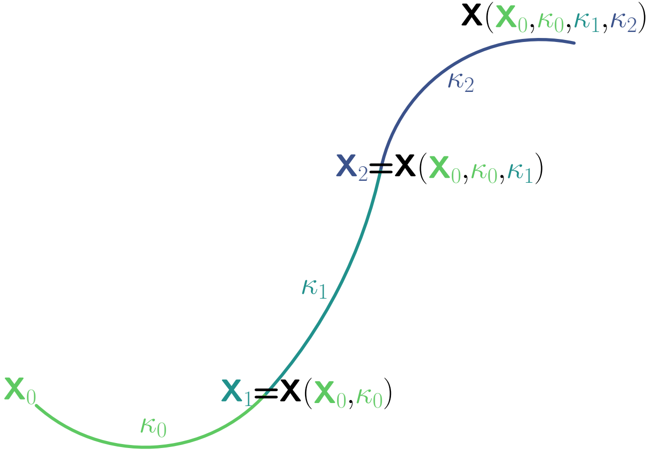
$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix}, \ \kappa_0, \ \boldsymbol{X}_1(\boldsymbol{X}_0, \kappa_0), \ \kappa_1, \ \boldsymbol{X}_2(\boldsymbol{X}_0, \kappa_0, \kappa_1), \ \kappa_2,$$

Chained super-circle model



$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix},\ \kappa_0,\ \boldsymbol{X}_1(\boldsymbol{X}_0, \kappa_0),\ \kappa_1,\ \boldsymbol{X}_2(\boldsymbol{X}_0, \kappa_0, \kappa_1),\ \kappa_2,\ \ldots$$

Chained super-circle model



$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix},\ \kappa_0,\ \boldsymbol{X}_1(\boldsymbol{X}_0, \kappa_0),\ \kappa_1,\ \boldsymbol{X}_2(\boldsymbol{X}_0, \kappa_0, \kappa_1),\ \kappa_2,\ \ldots$$

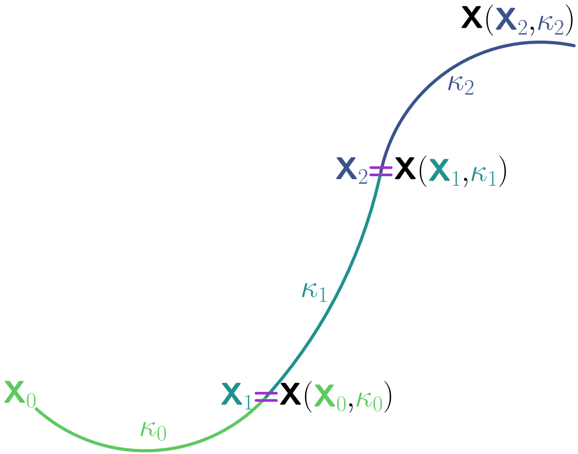
$$\boldsymbol{M} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

$$\boldsymbol{M}\boldsymbol{\Delta} = \boldsymbol{f}$$

$$(n+3) \times (n+3)$$

dense system

Mixed super-circle model

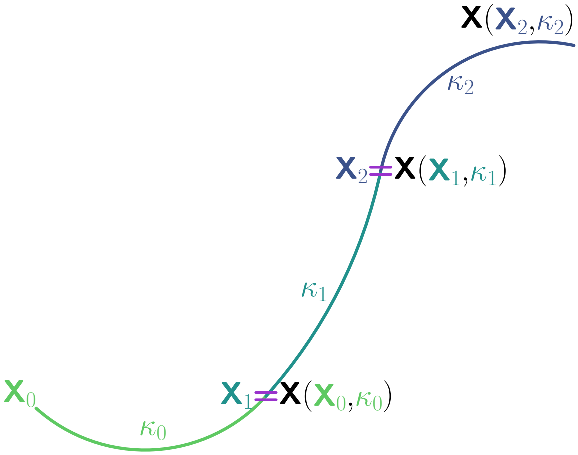


- Constraints of the form

$$g(\boldsymbol{X}) = 0.$$

$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix},\ \kappa_0,\ \boldsymbol{X}_1 = \begin{pmatrix} \boldsymbol{x}_1 \\ \theta_1 \end{pmatrix},\ \kappa_1,\ \boldsymbol{X}_2 = \begin{pmatrix} \boldsymbol{x}_2 \\ \theta_2 \end{pmatrix},\ \kappa_2,\ \ldots$$

Mixed super-circle model



- Constraints of the form

$$g(\boldsymbol{X}) = 0.$$

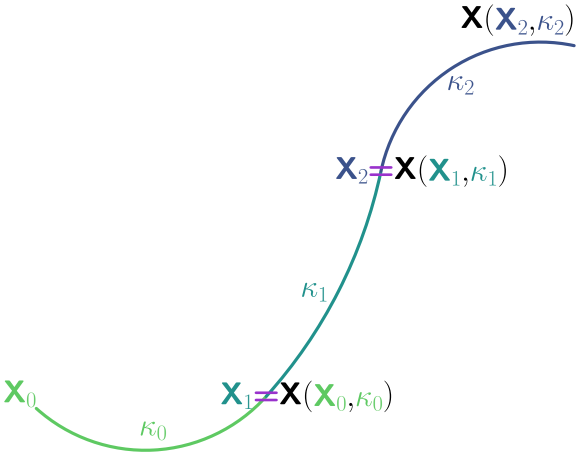
- Linearised in the form

$$\boldsymbol{C}\boldsymbol{\Delta} + \boldsymbol{c} = 0.$$

$\boldsymbol{C}$ sparse-block matrix.

$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix}, \ \kappa_0, \ \boldsymbol{X}_1 = \begin{pmatrix} \boldsymbol{x}_1 \\ \theta_1 \end{pmatrix}, \ \kappa_1, \ \boldsymbol{X}_2 = \begin{pmatrix} \boldsymbol{x}_2 \\ \theta_2 \end{pmatrix}, \ \kappa_2, \ \ldots$$

Mixed super-circle model



$$\boldsymbol{X}_0 = \begin{pmatrix} \boldsymbol{x}_0 \\ \theta_0 \end{pmatrix}, \kappa_0, \boldsymbol{X}_1 = \begin{pmatrix} \boldsymbol{x}_1 \\ \theta_1 \end{pmatrix}, \kappa_1, \boldsymbol{X}_2 = \begin{pmatrix} \boldsymbol{x}_2 \\ \theta_2 \end{pmatrix}, \kappa_2, \ldots$$

- Constraints of the form

$$g(\boldsymbol{X}) = 0.$$

- Linearised in the form

$$\boldsymbol{C}\boldsymbol{\Delta} + \boldsymbol{c} = 0.$$

$\boldsymbol{C}$ sparse-block matrix.

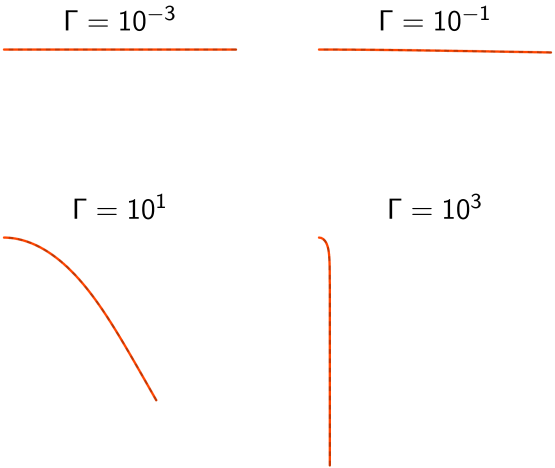
- Sparse system overall

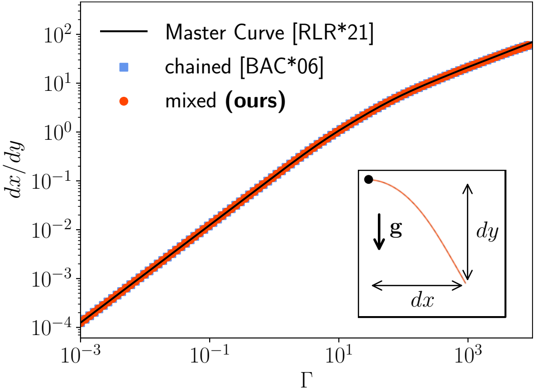
$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{C}^T \\ \boldsymbol{C} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\Delta} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ -\boldsymbol{c} \end{pmatrix}$$

$\boldsymbol{M} = \mathrm{diag}(\boldsymbol{M}_i)$
$7n \times 7n$ sparse system

Cantilever test: Setup and master curve

The tip deflection $\frac{dx}{dy}$ of the rod depends only on the mass over stiffness ratio Γ.



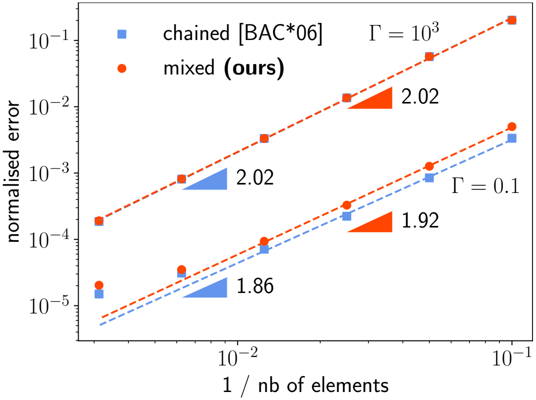


Cantilever test: Convergence rate

The relative error to the master curve depends on the number of elements in the discretisation.

$\Gamma = 10^{-1}$ $\Gamma = 10^{3}$

10 elements

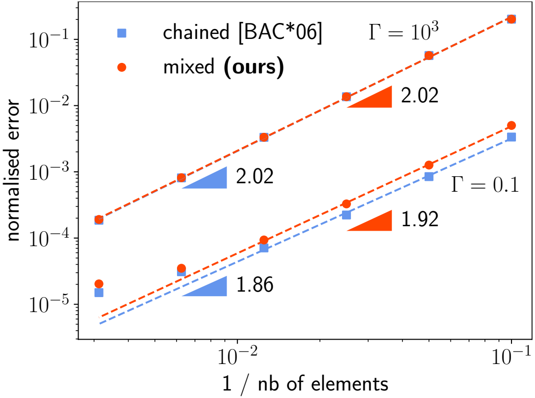


Cantilever test: Convergence rate

The relative error to the master curve depends on the number of elements in the discretisation.

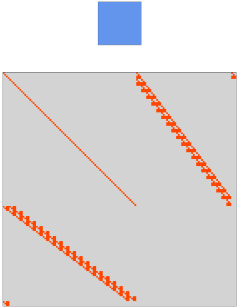
$\Gamma = 10^{-1}$ $\Gamma = 10^{3}$

320 elements

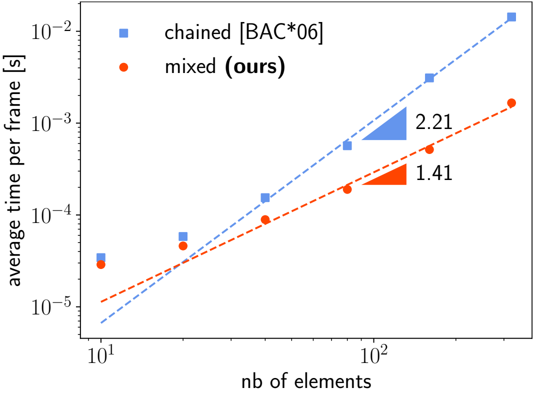


Cantilever test: Complexity time

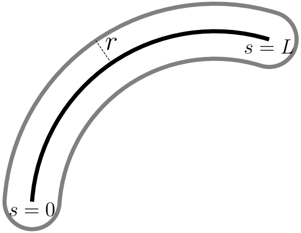
The computation time increases with the number of elements in the discretisation.



- chained: inverse small dense matrix
- mixed: inverse big sparse matrix

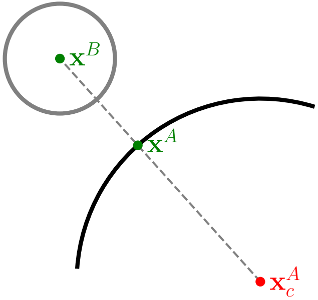


High order contact detection



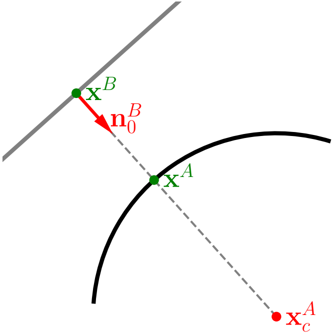
- **Curvilinear** elements,
- total length L,
- uniform width (or radius) r.

High order contact detection



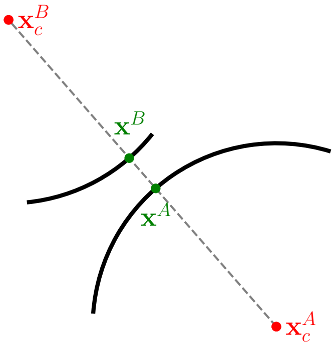
- **Curvilinear** elements,
- total length L,
- uniform width (or radius) r.
- Analytic solutions for
  - $\rightarrow$ disk,

High order contact detection



- **Curvilinear** elements,
- total length L,
- uniform width (or radius) r.
- Analytic solutions for
  - → disk,
  - → plane,

High order contact detection



- **Curvilinear** elements,
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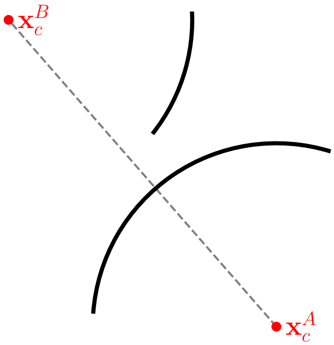
- Analytic solutions for
  - → disk,
  - → plane,
  - → arc.

High order contact detection



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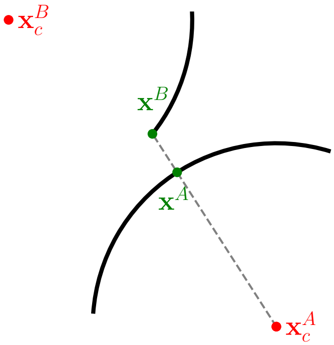
High order contact detection



- **Curvilinear** elements,
- total length L,
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- Analytic solutions for
  - → disk,
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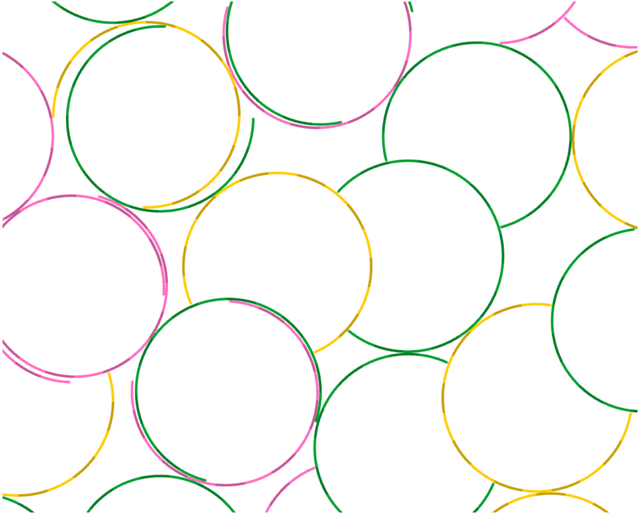
- If the intersection lies outside the element, there is a finite number of tests to do.

High order contact detection



- **Curvilinear** elements,
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- Analytic solutions for
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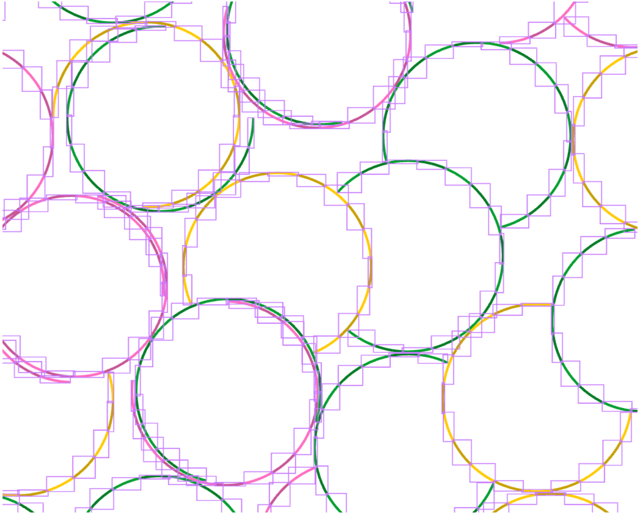
- If the intersection lies outside the element, there is a finite number of tests to do.
- Coupled with a **broad phase**.

High order contact detection



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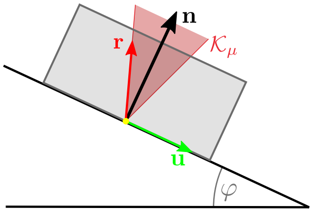
Contact solver

Discrete Frictional Complementarity Problem
[Cadoux 2009]

$$\begin{cases} \boldsymbol{M}\boldsymbol{\Delta} & = \boldsymbol{f} + \boldsymbol{C}^T\boldsymbol{\lambda} + \boldsymbol{H}^T\boldsymbol{r} \\ \boldsymbol{C}\boldsymbol{\Delta} + \boldsymbol{c} & = 0 \\ \boldsymbol{H}\boldsymbol{\Delta} + \boldsymbol{h} & = \boldsymbol{u} \\ (\boldsymbol{u}, \boldsymbol{r}) & \in \mathcal{C}(\mu) \end{cases}$$



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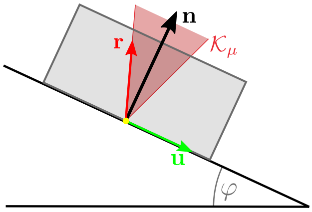
Consecutive Schur reductions to simplify to

$$(\boldsymbol{W}\boldsymbol{r} + \boldsymbol{w}, \boldsymbol{r}) \in \mathcal{C}(\mu)$$



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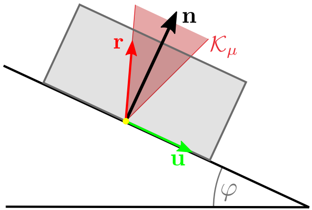

Projected Gauss-Seidel

$\boldsymbol{r} \leftarrow \boldsymbol{0}$
for $n = 1, \ldots, step\_max$ **do**
  for each contact i **do**
    $\boldsymbol{u}^i \leftarrow \frac{(\boldsymbol{W}\boldsymbol{r}+\boldsymbol{w})^i}{\sqrt{{\boldsymbol{W}^{i,i}_{n,n}}^2 + {\boldsymbol{W}^{i,i}_{t,t}}^2}}$
    $\boldsymbol{u}^i_n \leftarrow \boldsymbol{u}^i_n + \mu^i |\boldsymbol{u}^i_t|$ ▷ [DF98]
    $\boldsymbol{r}^i \leftarrow \text{project}(\boldsymbol{r}^i - \boldsymbol{u}^i, \ \mathcal{K}_{\mu^i})$

Contact solver

Discrete Frictional Complementarity Problem [Cadoux 2009]

$$\begin{cases} \boldsymbol{M}\boldsymbol{\Delta} &= \boldsymbol{f} + \boldsymbol{C}^T\boldsymbol{\lambda} + \boldsymbol{H}^T\boldsymbol{r} \\ \boldsymbol{C}\boldsymbol{\Delta} + \boldsymbol{c} &= 0 \\ \boldsymbol{H}\boldsymbol{\Delta} + \boldsymbol{h} &= \boldsymbol{u} \\ (\boldsymbol{u}, \boldsymbol{r}) &\in \mathcal{C}(\mu) \end{cases}$$

Consecutive Schur reductions to simplify to

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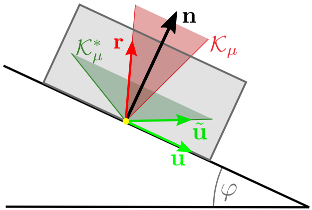

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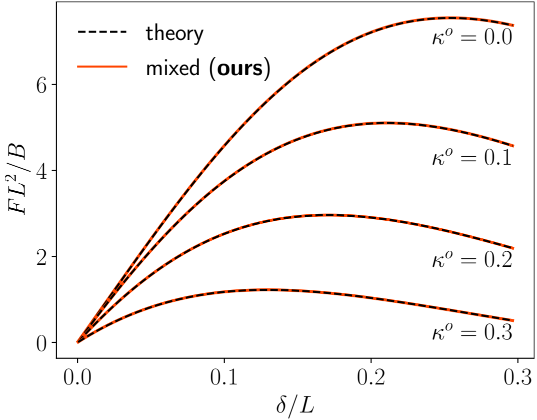
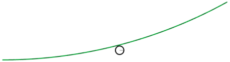
    $\boldsymbol{u}_n^i \leftarrow \boldsymbol{u}_n^i + \mu^i |\boldsymbol{u}_t^i|$ ▷ [DF98]
    $\boldsymbol{r}^i \leftarrow \text{project}(\boldsymbol{r}^i - \boldsymbol{u}^i,\ \mathcal{K}_{\mu^i})$

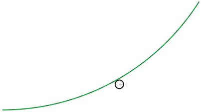
Curved three point bending experiment

$\kappa^o = 0.0$

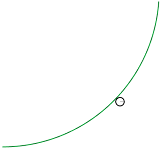




$\kappa^o = 0.1$



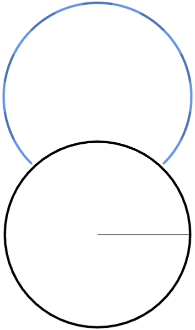
$\kappa^o = 0.2$



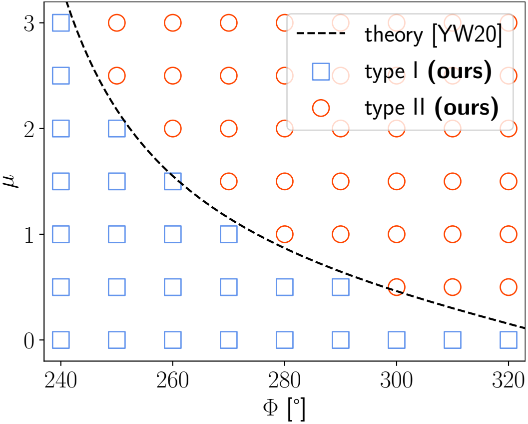
$\kappa^o = 0.3$

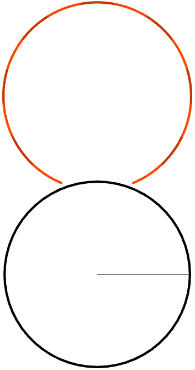
[Crespel et al. 2024]

Ring on cylinder snap-fit



Type I

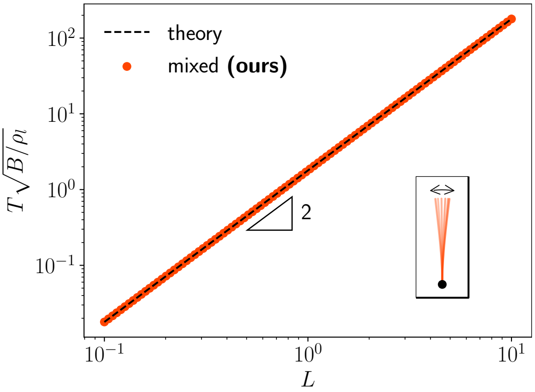




Type II

[Yoshida & Wada 2020]

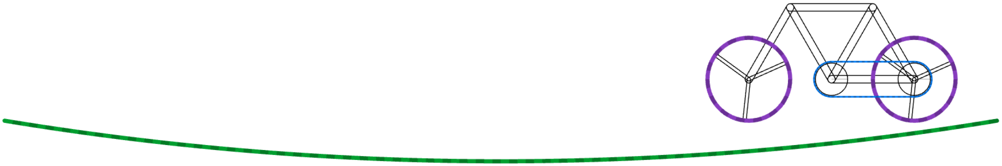
Flexural vibration test

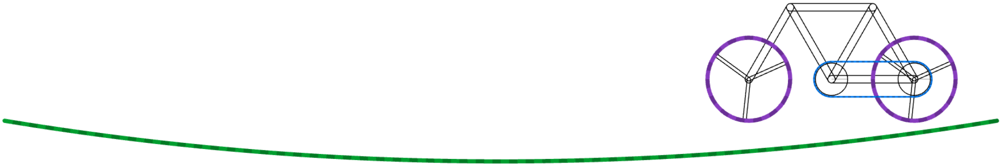


$L = 1.0$

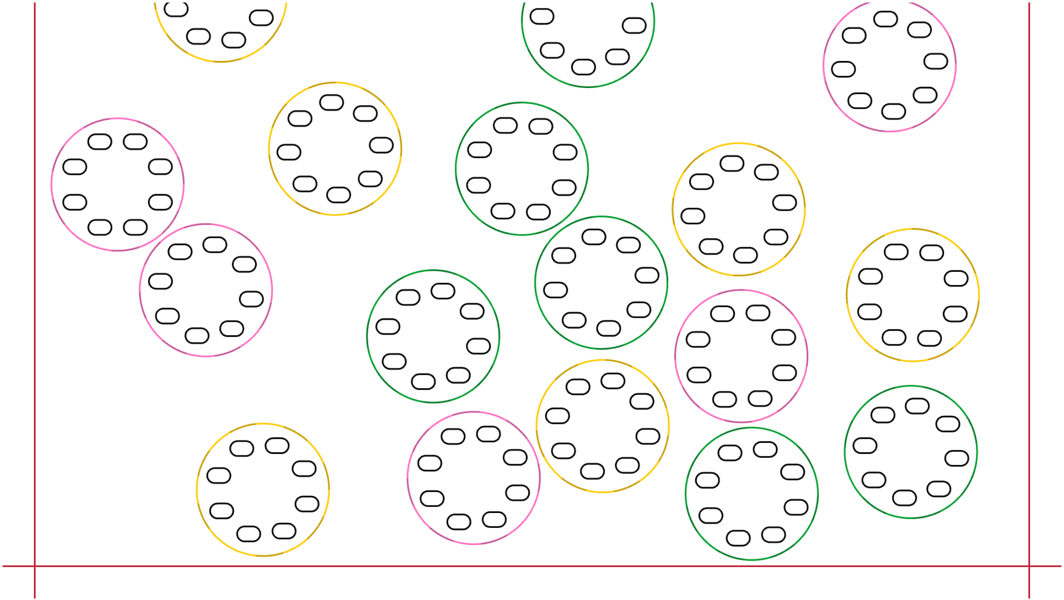
$L = 2.0$

Bicycle on a rope

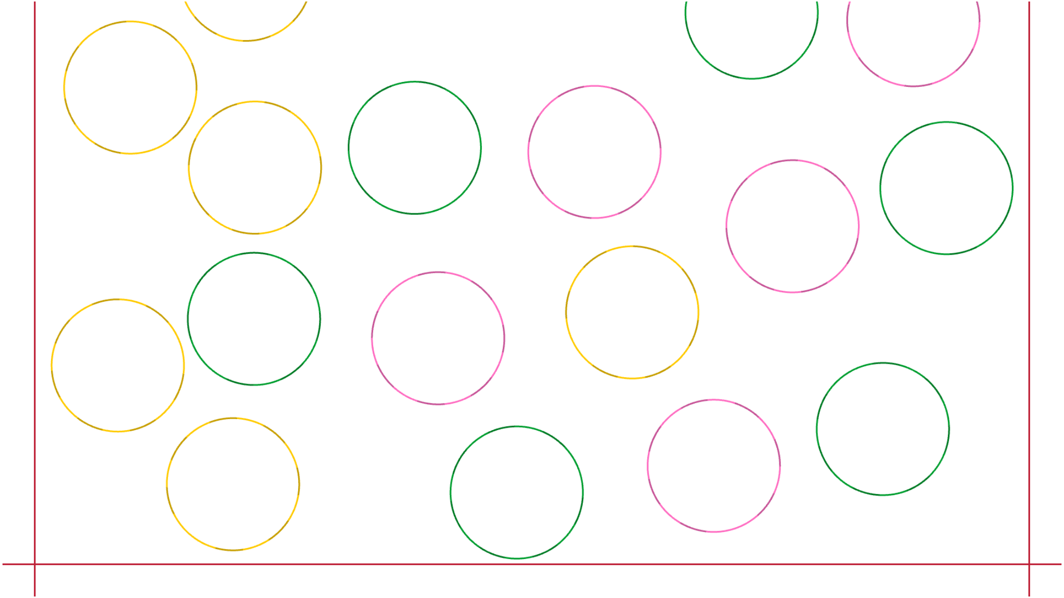




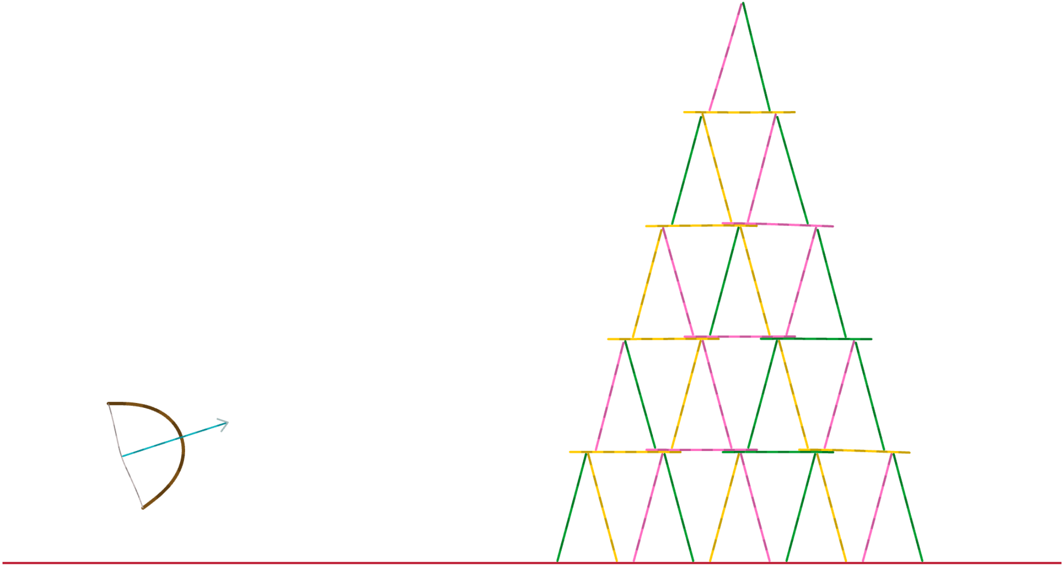
Stack of shells $\Gamma = 10$



Stack of shells $\Gamma = 100$



Archery



Source code

https://gitlab.inria.fr/elan-public-code/circonflex

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